Analysis 1B — Integral Test

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# Introduction

We’ve reached the end of the course! However, despite their prominance in Analysis 1A, we didn’t really say much about infinite series. So, to finish off this semester, I wanted to give you a test for series convergence which we can develop using the theory of integration. This is non-examinable, but the method might come in useful for future courses! Furthermore, the examples here may serve as good practice for unseen exam questions.

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# The Test

Theorem 1.1 (Integral Test for Series)

Suppose is a real sequence. Suppose also that a function is positive and decreasing on and that for all . Then, the series converges if and only if the limit

exists.

Proof.

Note that the existence of is equivalent (by linearity of integration) to the convergence of the series

Now, since is decreasing, for each , we can use the subdivision of the intervals to find

Applying the comparison test to the left hand side of (\*) shows that if exists, then (and hence ) also exists. This proves that

Finally, applying the comparison test to the right hand side of (\*) shows that if exists then also exists. This proves the remaining statement, i.e.

□

Note that we can replace with any in this theorem (such as in the lower series/integral limit), and the resulting modified version of the test still works.

# Example

Providing a result without any practical uses is a bit pointless. So here’s an example of this theorem in action! The question(s) here are taken from the textbook *‘Calculus’* by Michael Spivak.

Question:

1. Show that exists, by considering the series
2. Show that
3. converges, by using the integral test. *Hint: use an appropriate substitution and part (a)*.
4. Show that
5. diverges, by using the integral test. *Hint: Use the same substitution as in part (b), and show directly that the resulting integral diverges.*

## Solutions

Solution (Part a).

Firstly, setting , we have

Taking , the algebra of limits gives that as

so by d’Alembert’s ratio test, the series is convergent.

Now, define by Note that is strictly decreasing on and for each , . Hence, by the integral test, the integral exists, as required.

Solution (Part b).

Consider the function given by

Setting , we find that

which exists by part a). Now, for all , we have that . Also, by the chain rule, we find that on ,

which is always negative, so is decreasing on Hence, by the integral test, we find that the series

converges.

Solution (Part c).

Consider the function given by

By differentiating, we can show that is strictly decreasing on , so we can apply the integral test to this function.

Now, setting we have that (if it exists),

By rules of exponentials, we can rewrite the integrand as

Writing , we know that (by e.g. the growth factor test)

So by the definition of convergence at (see Problem Sheet 3), we know that such that for all ,

Rearranging and multiplying by , we find

from which raising everything to the power of yields

Finally, by properties of the integral, we have that and large enough ,

Using the fundamental theorem of calculus, we can evaluate the right hand integral to obtain

This right hand side of this inequality diverges as , and since is finite, the original improper integral also diverges. Hence, by the integral test, the series

diverges.